Canonical and micro-canonical typical entanglement of continuous variable systems

2007 J. Phys. A: Math. Theor. 40 9551

(http://iopscience.iop.org/1751-8121/40/31/027)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 9551–9576 doi:10.1088/1751-8113/40/31/027

Canonical and micro-canonical typical entanglement of continuous variable systems

A Serafini[1,2], **O C O Dahlsten**[1,2], **D Gross**[1,2] **and M B Plenio**[1,2]

[1] Institute for Mathematical Sciences, 53 Prince's Gate, Imperial College London, London SW7 2PG, UK
[2] QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, UK

Received 10 January 2007, in final form 13 June 2007
Published 19 July 2007
Online at stacks.iop.org/JPhysA/40/9551

Abstract

We present a framework, compliant with the general canonical principle of statistical mechanics, to define measures on the set of pure Gaussian states of continuous variable systems. Within such a framework, we define two specific measures, referred to as 'micro-canonical' and 'canonical', and apply them to study systematically the statistical properties of the bipartite entanglement of n-mode pure Gaussian states at, respectively, given maximal energy and given temperature. We prove the 'concentration of measure' around a finite average, occurring for the entanglement in the thermodynamical limit in both the canonical and the micro-canonical approach. For finite n, we determine analytically the average and standard deviation of the entanglement (as quantified by the reduced purity) between one mode and all the other modes. Furthermore, we numerically investigate more general situations, clearly showing that the onset of the concentration of measure already occurs at relatively small n.

PACS numbers: 03.67.Mn, 05.70.−a

(Some figures in this article are in colour only in the electronic version)

1. Typical entanglement in quantum information theory

Due to the exponentially increasing complexity of the Hilbert spaces of multiple constituents, a complete theoretical characterization of the entanglement of general quantum systems of many particles turns out to be a daunting task [1]. A viable approach towards such a characterization consists in focusing on the 'typical', *statistical* properties of the quantum correlations of multipartite systems, when the states of the system are assumed to be distributed according to a particular 'measure'. This strategy is firstly aimed at simplifying the problem at hand by restricting attention on the typical (and thus, in a sense to be precisely specified in the following, 'overwhelmingly likely') features of the entanglement of a system whose state,

resulting from a random process, is apt to be described by the chosen measures. Furthermore, this kind of analysis is able to shed light on the general properties of the entanglement of physical systems.

For finite-dimensional quantum systems, a natural, 'uniform' measure on pure states emerges from the 'Haar' measure of the unitary group (i.e., from the left- and right-invariant measure under application of any unitary transformation), whose elements allow one to retrieve any state when applied to another given starting pure state. On such grounds, a well-defined typical entanglement of finite-dimensional systems can be addressed and analysed. Original studies in this direction were undertaken well before the development of the formal theory of entanglement developed in quantum information science [1]: in 1978, Lubkin considered the expected entropy of a subsystem when picking pure quantum states at random from the uniform measure [2]. Let us recall that the von Neumann entropy $S = -\mathrm{tr}\rho \ln \rho$ of a subsystem in state ρ properly quantifies, for globally pure states, the entanglement between the subsystem and the remainder of the system. Lubkin showed that one expects this quantity to be nearly maximal. Pagels and Lloyd arrived later at the same qualitative conclusions, following an independent line of thought [3]. Their work was expanded by Page, who conjectured an exact formula for the average entropy of a subsystem $S_{m,n}$ [4], reading

$$S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}, \tag{1}$$

for a quantum system of Hilbert space dimension mn in a random pure state, and a subsystem of dimension $m \leqslant n$. This relation was later proven by Foong and Kanno [5].

This general line of enquiry was revisited and considerably extended in the setting of quantum information theory by Hayden, Leung and Winter in [6], where they extensively studied the 'concentration of measure' around the average of the entanglement probability distribution with increasing n. They also pointed out that this study may provide a way of simplifying the theory of entanglement which contains a plethora of locally inequivalent classes. In other words, as already mentioned, restricting statements to the 'typical entanglement' allows one to ignore several unessential complications. Recent results on the physical interpretation of Page's conjecture can be found in [7, 8], where it is proven that a circuit of elementary quantum gates on a quantum circuit is expected to maximally entangle the state to a fixed arbitrary accuracy, within a number of gates that grows only polynomially in the number of qubits of the register.

A further simplification in the analysis of the entanglement can be achieved by considering the typical entanglement of 'particularly relevant' (according to the specific problem at hand) subsets of states. For example, it has been found that 'stabilizer states' (a countable set of states playing a central role in quantum error correcting codes) are also typically maximally entangled [9, 10], similarly to the set of all states. Such investigations are interesting *per se*, as they unveil the potential and limitations hidden in the adoption of restricted classes of states and, furthermore, provide us with a more detailed understanding of the entanglement properties of the total state space.

Here we expand these considerations into the realm of continuous variables, i.e. of quantum systems described by pairs of canonically conjugated observables with continuous spectra. Such systems, ranging from motional degrees of freedom of particles in first quantization to bosonic fields in second quantization, are ubiquitous to all areas of quantum physics, being prominent in quantum optics (as they embody the light field in second quantization), atomic physics (notably, in the description of atomic ensembles), quantum field theory (as they encompass any bosonic field), in addition to their crucial role in molecular and atomic physics.

Quantum systems described by operators with continuous spectra live in infinite-dimensional Hilbert spaces. Therefore, a first naive try in this direction could be to take the infinite-dimensional limit of equation (1). Page showed that equation (1) implies that $S_{m,n} \simeq \ln m - \frac{m}{2n}$ for $1 \ll m \leqslant n$. Under that restriction, and noting that the maximal entanglement is $\ln m$, we can then make the observation that the ratio of the entanglement average to the maximum tends to unity as $n \longrightarrow \infty$, but that the two quantities both diverge logarithmically. From this perspective, the entanglement could be said to be typically infinite in the continuous variable setting. Even though mathematically reasonable (in the limit's sense), this statement is definitely questionable, physically and practically. In fact, in any practical situation, one will deal with a finite total energy or with finite 'temperatures' (both quantities will be defined precisely in our treatment), whereas infinitely entangled states require an infinite energy to be created.

In this paper, we shall restrict our attention on pure Gaussian states whose typical entanglement we shall study under two different measures ('canonical' and 'micro-canonical'), both inspired by arguments of thermodynamical natures, but apt to describe different situations. The previously mentioned divergences still potentially emerge in the Gaussian setting, essentially because the symplectic group (which corresponds to second-order operations and 'generates' any pure Gaussian state from the vacuum) is non-compact, and thus allow one to reach infinitely entangled states with infinite energy. The non-compactness of the symplectic group is also the reason which prevents one from straightforwardly defining normalizable measures on the set of Gaussian states. After having highlighted in detail such difficulties, we will present here a general framework to circumvent them and define proper, physical measures on the set of pure Gaussian states. The first ingredient we will make use of will be the 'Euler decomposition' of a symplectic operation S, reading

$$S = O' Z O,$$

where O and O' are orthogonal transformations (which do not affect the energy of the system), while Z is a non-energy preserving product of local squeezings. Such a decomposition will allow us to distinguish between 'well-behaved' compact parameters, contained in O and O', and the 'problematic' squeezing parameters of Z, which are responsible of the non-compact nature of the group and from which all the divergences originate. Eventually, such divergences will be tamed by introducing proper prescriptions which will affect the energy of the system, by imposing either a sharp upper bound (for the micro-canonical measure) or an exponentially decaying (for the canonical measure) distribution of energies (in this respect, see also [11] for a detailed discussion of the impact of similar constraints on entanglement measures). These prescriptions will be motivated on thermodynamical grounds, by requesting our distributions on the set of states to comply with the 'general canonical principle' [14], stating that reductions of large systems should always be in thermal states (see page 8 for a precise formulation). The compliance with the principle renders the measures we will define particularly apt to describe situations at thermodynamical equilibrium, when the principle always holds. Actually, the general canonical principle imposes restrictions on the possible measures only in the asymptotic limit of infinite number of constituents: for finite number of degrees of freedom there could be other possible measures, within our general framework, than the micro-canonical and canonical instances here considered. However, also for a finite number of constituents, these two measures permit us to answer the questions we are ultimately interested in: what is the typical entanglement of Gaussian states *for, respectively, given energy and temperature?* We will show that such measures imply the occurrence of a *finite* typical entanglement in the limit of an infinite number of total constituents. Moreover, even for a finite number of total degrees of freedom and finite upper bound to the energy—entailing the

existence of a finite maximal entanglement—the typical entanglement concentrates around a value which is well distant from the allowed maximum.

Note that Gaussian states are certainly the most prominent class of states not only in quantum information with continuous variables, but also, more broadly, in quantum optics, as they can be generated and manipulated with relative ease (even as highly entangled states [15–17]), can be used for the implementation of quantum communication and information protocols [18] and serve as a powerful testing ground for the theoretical characterization of entanglement properties [19].

The micro-canonical measure, which we will apply here to the study of the typical entanglement, has been already employed in the analysis of the quantum teleportation of Gaussian states with generic second moments [20]. In particular, the micro-canonical average quantum fidelity and a corresponding 'classical threshold' have been evaluated for the teleportation of states with null first moments and arbitrary second moments under the standard continuous variable teleportation protocol (see, e.g., [18] for a description of the scheme).

Let us finally mention that a definition of micro-canonical average entanglement has been very recently addressed for finite-dimensional systems as well [21], with a major emphasis on the possibility of reducing time-averages to ensemble-averages.

This paper is organized as follows. In section 2, we review some preliminary facts about Gaussian states and set the notation. In section 3, we review the definition of the micro-canonical measure on pure Gaussian states, already introduced in [20], completing our previous analysis with the inclusion of comments and mathematical details previously omitted, and we extend the existing framework to encompass a 'canonical' measure as well. Section 4 contains a rigorous proof of the 'concentration of measure', common to the two measures introduced here, i.e. of the fact that the entanglement probability distribution concentrates in the thermodynamical limit, around a finite 'thermal' average, away from the allowed maximum. Even though this result had been anticipated in [20], this proof is original and adds further insight into the matter. Section 5 presents a detailed study about the typical entanglement of pure Gaussian states with a finite number of degrees of freedom, where both analytical findings and numerical evidences are reported. Conclusions and outlook are found in section 6. Three appendices complement the work, one of which (appendix B) contains the most technical steps needed to prove the concentration of measure, while in appendix A a specific Baker–Campbell–Hausdorff relation is derived, and in appendix C a derivation of the expression of the maximal entanglement for given energy is presented.

2. Preliminaries

We consider bosonic continuous variable (CV) quantum-mechanical systems described by n pairs of canonically conjugated operators $\{\hat{x}_j, \hat{p}_j\}$ with continuous spectra, like motional degrees of freedom of particles in first quantization or bosonic field operators in second quantization. Grouping the canonical operators together in the vector $\hat{R} = (\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n)^{\mathsf{T}}$ allows us to express the canonical commutation relations (CCR) as $[\hat{R}_j, \hat{R}_k] = 2i\Omega_{jk}$, where the 'symplectic form' Ω is defined as

$$\Omega = \begin{pmatrix} 0_n & \mathbb{1}_n \\ -\mathbb{1}_n & 0_n \end{pmatrix},$$

0_n and $\mathbb{1}_n$ standing for the null and identity matrix in dimension n.

Any state of an n-mode CV system is described by a positive, trace-class operator ϱ. For any state ϱ, let us define the $2n$-dimensional vector of the expectation values ('first moments') of the canonical operators R (with entries R_j) as

$$R_j \equiv \mathrm{tr}\,[\hat{R}_j \varrho]$$

and the $2n \times 2n$ matrix of second moments, or 'covariance matrix' (CM), $\boldsymbol{\sigma}$ (with entries $\sigma_{i,j}$) as

$$\sigma_{i,j} \equiv \mathrm{tr}\,[\{\hat{R}_i, \hat{R}_j\}\varrho]/2 - \mathrm{tr}\,[\hat{R}_i \varrho]\,\mathrm{tr}\,[\hat{R}_j \varrho].$$

Also, throughout the paper, we will refer to the 'energy' of a state ϱ as to the expectation value of the operator $\hat{H}_0 = \sum_{j=1}^{n} \left(\hat{x}_j^2 + \hat{p}_j^2\right)$. This definition corresponds to the energy of a free electromagnetic field in the optical scenario (and to decoupled oscillators in the general case). In our convention, as determined by the factor 2 appearing in the CCR, the vacuum of a single mode has covariance matrix equal to the identity (thus simplifying significantly several expressions), with energy 2 (the adopted energy unit is $\hbar\omega/4$ for a mode of frequency ω). The energy is determined by first and second moments according to

$$\mathrm{tr}\,(\varrho \hat{H}_0) = \mathrm{tr}(\boldsymbol{\sigma}) + \|R\|^2, \tag{2}$$

where $\|R\|$ is the usual Euclidean norm of the vector R.

Gaussian states are the states with Gaussian characteristic functions and quasi-probability distributions, defined over a phase space analogous to that of classical Hamiltonian dynamics. As well known, a pure state $|\psi\rangle_G$ is Gaussian if and only if it can be obtained by transforming the vacuum $|0\rangle$ under an operation generated by a polynomial of the second order in the canonical operators. In formulae (up to a negligible global phase factor):

$$|\psi\rangle_G = \hat{G}_{A,b}|0\rangle \equiv \mathrm{e}^{\mathrm{i}(\hat{R}^{\mathsf{T}} A \hat{R} + \hat{R}^{\mathsf{T}} b)}|0\rangle, \tag{3}$$

where A and b are, respectively, a real $2n \times 2n$ matrix and a real $2n$-dimensional vector. Because of the CCR and of the unitarity of $\hat{G}_{A,b} \equiv \mathrm{e}^{\mathrm{i}(\hat{R}^{\mathsf{T}} A \hat{R} + \hat{R}^{\mathsf{T}} b)}$, A can be chosen symmetric, without loss of generality, while b is a generic real vector. b and A embody the first and second moments of the Gaussian state, thus completely determining it.

The unitary operator can always be rewritten as (see appendix A)

$$\hat{G}_{A,b} = \mathrm{e}^{\mathrm{i}(\hat{R}^{\mathsf{T}} A \hat{R})}\, \mathrm{e}^{\mathrm{i}(\hat{R}^{\mathsf{T}} M b)}, \tag{4}$$

for some matrix M (as shown in appendix A, $M = \Omega A^{-1}(\mathbb{1}_{2n} - \mathrm{e}^{4A\Omega})/4$ for invertible A's). First- and second-order operations can thus be generally 'decoupled'.

First-order operations correspond to local displacements in phase space. While such operations do not affect local entropies (and thus the entanglement) of multipartite states, they do affect the energy of the states, which will play a central role in what follows. Moreover, let us note that the group of these transformations is non-compact, being isomorphic to the Abelian $\mathbb{R}^{2n}$ under the addition composition rule. In the following, we will show how the first moments can be consistently incorporated in the presented framework. However, because their inclusion in the study of the statistical properties of the entanglement is just a technicality (adding no significant insight), we will set them to zero in the investigations to come.

As for second-order transformations, determined by the matrix A, they can be conveniently mapped into the group $Sp_{2n,\mathbb{R}}$ of real *symplectic transformations*, acting linearly on phase space (as second-order transformations acting on the Hilbert space make up the multi-valued *metaplectic* representation of the symplectic group [22]). Recall that a matrix S belongs to the symplectic group $Sp_{2n,\mathbb{R}}$ if and only if it preserves the antisymmetric form Ω: $S \in SL(2n,\mathbb{R}) : S \in Sp_{2n,\mathbb{R}} \Leftrightarrow S^{\mathsf{T}}\Omega S = \Omega$. Let us also recall that a symplectic transformation S acts by congruence on a covariance matrix $\boldsymbol{\sigma}$: $\boldsymbol{\sigma} \mapsto S^{\mathsf{T}}\boldsymbol{\sigma} S$. Of course,

symplectic transformations can in general affect both the entanglement and the energy of a state. The algebra of generators of the symplectic group comprises all the matrices that can be written as ΩJ, where J is some $2n \times 2n$ symmetric matrix [23] (in this notation, generators are not complexified, so that $S = \mathrm{e}^{\Omega J}$). Such generators do not have a definite symmetry (i.e., they are not necessarily symmetric or antisymmetric). Choosing a basis of the algebra such that each generator of the basis is either symmetric or antisymmetric, allows one to distinguish between a *compact* subgroup $K(n) = Sp_{2n,\mathbb{R}} \cap SO(2n)$ (spawned by antisymmetric generators) and a *non-compact* subgroup (arising from skew-symmetric generators). Note also that, since compact transformations are indeed orthogonal, they do not affect the energy of the states they act upon (explicitly, $\mathrm{tr}\,\boldsymbol{\sigma}$ and $\|R\|^2$ are both invariant under phase-space 'rotations').

Remarkably, the subgroup $K(n)$ is isomorphic to $U(n)$. Because this fact will be exploited throughout the whole work, we shall sketch its proof here. Let us define the transformation O by

$$O = \begin{pmatrix} X & Y \\ W & Z \end{pmatrix},$$

where X, Y, W and Z are $n \times n$ real matrices. It is straightforward to show that this transformation is symplectic and orthogonal if and only if $Z = X$, $W = -Y$, $X^{\mathsf{T}}X + Y^{\mathsf{T}}Y = \mathbb{1}$ and $X^{\mathsf{T}}Y - Y^{\mathsf{T}}X = 0$, so that

$$O = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}. \tag{5}$$

Now, let $U = X + \mathrm{i}Y$ be a matrix with real part X and imaginary part Y. The unitarity condition on U corresponds exactly to the previous two conditions on X and Y, thus demonstrating the existence of a bijective mapping from $U(n)$ to $K(n)$. The preservation of the composition rule can be straightforwardly checked out. Incidentally, this isomorphism implies that $K(n)$ has n^2-independent parameters.

Let us rephrase equation (3) to give a transparent parametrization of pure Gaussian states in phase-space terms, by considering the action of first- and second-order operations on the covariance matrix and on the first moments:

$$\boldsymbol{\sigma} = S^{\mathsf{T}} S, \qquad \text{with} \quad S \in Sp_{2n,\mathbb{R}}, \qquad R \in \mathbb{R}^{2n} \tag{6}$$

(recall that, in our units, the covariance matrix of the vacuum is the identity). Indeed, because of the peculiar nature of their characteristic functions, Gaussian states are completely determined by first and second moments of the canonical operators.

More generally, let us also recall that the covariance matrix $\boldsymbol{\Sigma}$ of any, pure or mixed, Gaussian state can be written as

$$\boldsymbol{\Sigma} = S^{\mathsf{T}} \boldsymbol{\nu} S, \tag{7}$$

where $S \in Sp_{2n,\mathbb{R}}$ and $\boldsymbol{\nu} = \mathrm{diag}(\nu_1, \ldots, \nu_n, \nu_1, \ldots, \nu_n)$ is a diagonal matrix with double-valued eigenvalues called the 'Williamson normal form' of $\boldsymbol{\Sigma}$ [24, 25] (corresponding to the normal-modes decomposition of positive definite quadratic Hamiltonians). The real quantities $\{\nu_j\}$ are referred to as the 'symplectic eigenvalues' of $\boldsymbol{\Sigma}$ and can be computed as the eigenvalues of the matrix $|\mathrm{i}\Omega\boldsymbol{\Sigma}|$. The symplectic eigenvalues hold all the information about the entropic quantities of the Gaussian state in question. In particular, the 'purity' $\mu \equiv \mathrm{tr}\,\varrho^2$ of the Gaussian state ϱ with CM $\boldsymbol{\Sigma}$ is determined as

$$\mu = 1 \Big/ \prod_{j=1}^{n} \nu_j = 1/\sqrt{\det \boldsymbol{\Sigma}}, \tag{8}$$

while the von Neumann entropy $S \equiv -\operatorname{tr}(\varrho \ln \varrho)$ reads

$$S = \sum_{j=1}^{n} h(\nu_j), \tag{9}$$

with the 'entropic function' $h(x)$ given by

$$h(x) = \frac{x+1}{2} \log_2 \left(\frac{x+1}{2} \right) - \frac{x-1}{2} \log_2 \left(\frac{x-1}{2} \right). \tag{10}$$

Being the eigenvalues of $|\mathrm{i}\Omega\boldsymbol{\Sigma}|$, the symplectic eigenvalues are continuously determined by 'symplectic invariants' (i.e. by quantities depending on the entries of the CM invariant under symplectic transformations), defined as the coefficients of the characteristic polynomial of such a matrix [27, 28]). This observation will be useful later on.

3. Measures on the set of pure Gaussian states

The present section is devoted to the definition of consistent measures on the set of pure Gaussian states, introducing a broad framework motivated by fundamental statistical arguments. We will review the construction of the 'micro-canonical measure', already introduced in [20], complementing such earlier studies with discussions and mathematical details. Furthermore, within the same general framework, we will present a novel, 'canonical' measure, thus extending our previous treatment.

A 'natural' measure to pick would be one invariant under the action of the operations which generate the set of states we are focusing on. In the previous section, we have analysed such a set of operations for pure Gaussian states, showing that it amounts to symplectic operations and displacements. One would thus be tempted to adopt the left- and right-invariant measure (i.e., the *Haar* measure) over such groups. Unfortunately, because the symplectic group is non-compact, the existence of a Haar measure on the whole group [from which a measure for pure Gaussian states could be derived via equation (6)] is not guaranteed. Notably, even if such a measure could be constructed, it would not be normalizable, giving rise to distributions with unbounded statistical moments. Moreover, some prescription has obviously to be introduced also to handle the first moments which are, in general, free to vary in the non-compact $\mathbb{R}^{2n}$ (note that the Euclidean volume is obviously invariant under left and right translations but is not a proper measure in the space of first moments because it is not normalizable, due to the non-compactness of $\mathbb{R}^{2n}$).

To cope with such difficulties we will introduce, in analogy with statistical mechanical treatments, assumptions on the energy of the states under examination, as defined in equation (2), which will constitute our 'privileged' physical observable (in a sense to be elucidated in the following). A proper structure to introduce a measure is inspired by a well-known decomposition of an arbitrary symplectic transformation S:

$$S = O' Z O, \tag{11}$$

where $O, O' \in K(n) = \mathrm{Sp}(2n, \mathbb{R}) \cap SO(2n)$ are orthogonal symplectic transformations, while

$$Z = Z' \oplus Z'^{-1}, \tag{12}$$

where Z' is a diagonal matrix with eigenvalues $z_j \geqslant 1 \, \forall \, j$. The set of such Z's forms a non-compact subgroup of $\mathrm{Sp}_{2n,\mathbb{R}}$ (corresponding to local squeezings), which will be denoted by $Z(n)$. The virtue of such a decomposition, known as 'Euler' (or 'Bloch–Messiah' [26]) decomposition, is immediately apparent, as it allows one to distinguish between the degrees of

freedom of the compact subgroup (essentially 'angles', ranging from 0 to 2π, which moreover do not affect the energy) and the degrees of freedom z_j's with non-compact domain. In particular, applying Euler decomposition to equation (6) leads to

$$\boldsymbol{\sigma} = O^{\mathsf{T}} Z^2 O. \tag{13}$$

Due to the rotational invariance of the vacuum in phase space, the number of free parameters of a pure Gaussian state of an n-mode system is thus $n^2 + 3n$ (taking the $2n$-independent first moments into account).

Quite naturally, we shall assume the n^2 parameters of the transformation O to be distributed according to the Haar measure of the compact group $K(n)$, carried over from $U(n)$ through the isomorphism described by equation (5). The set of such parameters will be compactly referred to as ϑ, while the corresponding Haar measure will be denoted by $\mathrm{d}\mu_H(\vartheta)$.

We have thus identified the variables which parametrize an arbitrary pure Gaussian state and imposed a distribution on a subset of them. A 'natural' measure has yet to emerge for the non-compact variables $\{z_j\}$ and $\{R_j\}$. As a first physical assumption, we will restrict to states for which the expectation value of the free field Hamiltonian $\hat{H}_0 = \sum_{j=1}^{n} \left(\hat{x}_j^2 + \hat{p}_j^2\right)$ is bounded from above, such that $\mathrm{tr}\,(\boldsymbol{\sigma}) + \|R\|^2 = \sum_{j=1}^{n} \left(z_j^2 + z_j^{-2} + |R_{2j-1}|^2 + |R_{2j}|^2\right) \leqslant E$. Note that, as it constrains only the *expectation value* of $\hat{H}_0$, such an assumption is not equivalent to a truncation of the Hilbert space, and does not reduce our setting to a finite-dimensional one: the support of the states we are considering is still infinite dimensional (as is the case for all Gaussian states). However, this proviso does allow one to tame the non-compact nature of the symplectic group and to define well-behaved measures on it, as we are about to see.

In order to further constrain the choice of measures on such variables we shall invoke now a fundamental statistical argument, which we shall refer to as general canonical principle, stating that

> Given a sufficiently small subsystem of the universe, almost every pure state of the universe is such that the subsystem is approximately in the 'canonical state' ϱ_c.

Such a principle, already well known in the context of quantum thermodynamics [12, 13], has also been recently reintroduced as the basis of a 'kinematical' formulation of statistical mechanics [14]. The 'canonical state' ϱ_c is, in our case, the local reduction of the global state with maximal entropy under the constraint of a maximal expectation value for the energy operator $\hat{H}_0$ given by E. This is exactly the definition of a 'thermal state' of the Hamiltonian $\hat{H}_0$, which is the reduction of a state that maximizes the entropy for a given energy E. The local canonical state ϱ_c of m modes is thus given by

$$\varrho_c = \frac{\mathrm{e}^{-\sum_{j=1}^{m} \left(\hat{x}_j^2 + \hat{p}_j^2\right)/T}}{\mathrm{tr}\left[\mathrm{e}^{-\sum_{j=1}^{m} (\hat{x}_j^2 + \hat{p}_j^2)/T}\right]}.$$

As implied by the foregoing equation, the state ϱ_c is Gaussian, because it can be expressed as the exponential of a quadratic polynomial in $\hat{x}_j$'s and p_j's. In particular, it can be shown that ϱ has vanishing first moments and CM $\boldsymbol{\sigma}_c = (1 + T/2)\mathbb{1}$ [29]. Here the 'temperature' T is defined by passage to the 'thermodynamical limit', that is for $n \to \infty$ and $E \to \infty$, $(E - 2n)/n \to T$ (assuming $k_B = 1$ for the Boltzmann constant). For ease of notation, in the following, the symbol $\simeq$ will imply that the equality holds in the thermodynamical limit, e.g.: $(E - 2n)/n \simeq T$. Note that we have required here the introduction of a maximum energy E or, alternatively, of a temperature T. In point of fact, such requirements are necessary to handle the non-compact part of the symplectic group. Note also that, in principle, two options are open in this respect, as one can introduce either an

upper bound to the energy or a temperature: essentially, these two distinct options characterize the two distinct approaches (micro-canonical and canonical) which we will detail in the next section.

Because the canonical state ϱ_c is Gaussian with vanishing first moments, the general canonical principle can be fully incorporated into our restricted (Gaussian) setting. As we have shown in [20], the compliance with the general canonical principle enforces a rather stringent restriction on the distribution of the non-compact variables. In particular, the general canonical principle is always satisfied if, in the thermodynamical limit, such variables are independent and identically distributed (i.i.d.)[3]. To keep our exposition lighter and more readable, we will not repeat here the technical derivation of this implication. Actually, it will be entirely subsumed, *a posteriori*, by the derivation of the 'concentration of measure' in section 4.

Before moving on with the definition of specific measures, let us comment on the first moments $\{R\}$. The general canonical principle imposes that, in the thermodynamical limit, their density of probability $p'(R)$ tend to a δ-distribution centred in 0 (in fact, the canonical state has vanishing first moments). A suitable example is $p'(R) = (n\lambda/\pi)^n \exp(-n\lambda\|R\|^2)$ for some constant λ (note that $\|R\|^2$ is the first moments' contribution to the energy). Let us remark that this class of distributions encompasses those usually adopted for coherent states, in the computation of classical teleportation thresholds [30, 31]. From now on, we will just set the first moments to zero: they can be coherently incorporated into our general picture following the recipe given above.

Let us now turn to second moments and sum up our line of thought so far: inspired by mathematical considerations and guided by physical arguments, we have defined a distribution for the 'compact' degrees of freedom ϑ (essentially, their Haar measure) and specified a prescription for the non-compact parameters z_j's, derived by upper bounding the energy of the states and by adopting a fundamental thermodynamical proviso, in the form of the 'general canonical principle'. More precisely, the measures satisfying such a principle will be apt to describe ensembles of systems at the thermodynamical equilibrium, where the principle always holds. In this sense, our treatment is 'thermodynamical' in essence, as it ultimately addresses ensembles resulting from randomizing processes which reproduce conditions of thermodynamical equilibrium.

Several choices are then possible, within the constraint of asymptotically i.i.d. variables dictated by the principle, to deal with the variables z_j's: generally, the choice of a specific measure will depend on the specific conditions of the system under exam. However, let us point out that this analysis allowed us to introduce a very general framework under which proper, normalized measures on the set of Gaussian states can be defined. To illustrate such a programme, we will now detail two possible choices, which will allow us to study the typical entanglement of Gaussian states at, respectively, given maximal energy and given temperature.

3.1. Micro-canonical measure

As a first approach, we will introduce a micro-canonical measure on the class of n-mode pure Gaussian states with an energy upper bounded by E. Note that such a restriction is essentially equivalent to *fixing* the total energy of $(n+1)$-mode states to $E+2$ (in fact, in the latter instance, the energy of the additional mode is not independent and merely 'makes up' to reach the fixed

[3] Even though the agreement between the two pairs of equations could allow for a more general choice, i.i.d. variables are by far the most convenient assumption for the sake of clarity of both mathematical description and physical interpretation (see [20]).

total amount). Notably, the two approaches are indistinguishable in the thermodynamical limit.

Note that the parameters $\{z_j\}$, whose distribution is left to define, determine the energy E_j pertaining to each *decoupled* mode j of the Euler decomposition, according to [see a equation (13)]

$$E_j = z_j^2 + \frac{1}{z_j^2}. \tag{14}$$

Here, we will assume a Lebesgue ('flat') measure for the local energies E_j's (uniquely determining the squeezings z_j's, as $z_j \geqslant 1$), inside the region $\Gamma_E = \{\boldsymbol{E} : |\boldsymbol{E}| \leqslant E\}$ bounded by the linear hypersurface of total energy E (here, $\boldsymbol{E} = (E_1, \ldots, E_n)$ denotes the vector of energies, with all positive entries, while $|\boldsymbol{E}| = \sum_{j=1}^n E_j$). More explicitly, denoting by $\mathrm{d}p_{mc}(\boldsymbol{E})$ the probability of the occurrence of the energies $\boldsymbol{E}$, one has

$$\begin{aligned} &\mathrm{d}p_{mc}(\boldsymbol{E}) = \mathcal{N}\mathrm{d}^n\boldsymbol{E} \equiv \mathcal{N}\mathrm{d}E_1 \cdots \mathrm{d}E_n \qquad \text{if} \quad \boldsymbol{E} \in \Gamma_E, \\ &\mathrm{d}p_{mc}(\boldsymbol{E}) = 0 \qquad \text{otherwise}, \end{aligned} \tag{15}$$

where $\mathcal{N}$ is a normalization constant equal to the inverse of the volume of Γ_E. Note that such a flat distribution is the one maximizing the entropy in the knowledge of the local energies of the decoupled modes. In this specific sense such variables have been privileged, on the basis of both mathematical (the Euler decomposition) and physical (analogy with the micro-canonical ensemble) grounds. Let us also mention that, as will become apparent in the next section, employing the variables $\{E_j\}$ leads to a remarkable simplification of the expression of the averages over the Haar measure of the compact subgroup. While purely formal, this aspect yields some significant insight into the privileged role of such variables in characterizing the statistical properties of physical quantities.

The micro-canonical average $Q_{mc}(E)$ over pure Gaussian states at maximal energy E of the quantity $Q(\boldsymbol{E}, \vartheta)$ determined by the second moments alone will thus be defined as

$$Q_{mc}(E) = \mathcal{N} \int \mathrm{d}\mu_H(\vartheta) \int_{\Gamma_E} \mathrm{d}\boldsymbol{E}\, Q(\boldsymbol{E}, \vartheta), \tag{16}$$

where the integration over the Haar measure is understood to be carried out over the whole compact domain of the variables ϑ. More explicitly, the integral over the energies E can be recast as

$$\int_{\Gamma_E} \mathrm{d}\boldsymbol{E} = \mathcal{N} \int_2^{E-2(n-1)} \mathrm{d}E_1 \cdots \int_2^{E-\sum_{j=1}^{n-1} E_j} \mathrm{d}E_n \tag{17}$$

(each energy is lower bounded by the vacuum energy, equal to 2 in the convention adopted here), determining the normalization as $\mathcal{N} = n!/(E-2n)^n$ and leading to a marginal density of probability $P_n(E_j, E)$ for each of the energies E_j given by

$$P_n(E_j, E) = \frac{n}{E-2n}\left(1 - \frac{E_j - 2}{E-2n}\right)^{n-1}. \tag{18}$$

Clearly, the energies E_j are not i.i.d. for finite n. However, as is apparent from equations (17) and (18), in the thermodynamical limit the upper integration extremum diverges for each E_j while, for the marginal probability distribution, one has $P_n(E_j, E) \simeq \mathrm{e}^{-\frac{E_j-2}{T}}/T$. In the thermodynamical limit, the decoupled energies are distributed according to independent Boltzmann distributions, with the parameter T playing the role of a temperature, in compliance with the equipartition theorem and equivalence of statistical ensembles of classical thermodynamics. This argument shows that the micro-canonical measure fulfils the general canonical principle. Also, it naturally brings us to introduce a 'canonical' measure on the set of pure Gaussian states.

3.2. Canonical measure

In the 'canonical' approach, we will assume for the energies $\boldsymbol{E}$ a probability distribution $\mathrm{d}p_c(\boldsymbol{E})$ reading

$$\mathrm{d}p_c(\boldsymbol{E}) = \frac{\mathrm{e}^{-(|\boldsymbol{E}|-2n)/T}}{T^n}\,\mathrm{d}\boldsymbol{E} = \prod_{j=1}^{n}\left(\frac{\mathrm{e}^{-(E_j-2)/T}}{T}\,\mathrm{d}E_j\right), \tag{19}$$

introducing a 'temperature' T. This distribution maximizes the entropy on the knowledge of the continuous variables E_j's for given average total energy E_{av}, such that $nT = E_{\mathrm{av}}$ (the latter relation is easily derived by applying Lagrange multipliers).

The 'canonical' average $Q_c(T)$ over pure Gaussian states at temperature T of the quantity $Q(E, \vartheta)$ determined by the second moments alone will thus be defined as

$$Q_c(T) = \int \mathrm{d}\mu_H(\vartheta) \int \frac{\mathrm{e}^{-(|\boldsymbol{E}|-2n)/T}}{T^n}\,\mathrm{d}\boldsymbol{E}\,Q(\boldsymbol{E}, \vartheta), \tag{20}$$

where the integration over the energies is understood to be carried out over the whole allowed domain ($E_j \geqslant 2\,\forall\, j$).

As already elucidated, the micro-canonical and canonical approaches coincide in the thermodynamical limit, as one should expect in analogy with the indistinguishability of the classical statistical ensembles in the thermodynamical limit.

Let us point out that the canonical distribution of states we have defined in this subsection results as the stationary distribution of states under the application of a randomizing process '*á la* Metropolis', where any initial pure Gaussian state undergoes compact single-mode symplectic operations ('phase-plates', in the quantum optical terminology) with random optical phase, fixed two-mode compact operations ('beam-splitters') between two random modes, and random single-mode squeezings, such that the squeezing operation is always accepted if it decreases the energy and probabilistically accepted (with Boltzmann weight) if it increases the energy. This provides an interpretation of the proposed measure in terms of random circuits.

4. Concentration of measure

In the present section, we will study the statistical properties of the entanglement of pure Gaussian states in the thermodynamical limit under the measures introduced in the previous section. More specifically, we shall focus on the behaviour of the entanglement of a subsystem of m modes (as quantified by the von Neumann entropy of the reduction describing such a subsystem), keeping m fixed and letting the total number of modes $n \to \infty$. As we have seen, the two measures coincide in this limit (when, in the micro-canonical treatment, the energy—notably the other extensive quantity in play—diverges as well). It will thus suffice to consider the canonical measure, and the micro-canonical averages will be retrieved upon identifying $T \equiv (E - 2n)/n$. In this section, the shorthand notation $\overline{x}$ will stand for the average of the quantity x with respect to the canonical state-space measure. Under the previous assumptions, we will determine the average asymptotic entanglement and prove that the variance of the entanglement tends to zero in the thermodynamical limit. The latter property, which had been previewed in [20], will be referred to as 'concentration of measure'.

Let us first recall that the von Neumann entropy S of the m-mode reduction is determined by the local symplectic eigenvalues $\{\nu_j, \text{for}\, 1 \leqslant j \leqslant m\}$ of the reduced m-mode CM γ according to

$$S = \sum_{j=1}^{m} h(\nu_j), \tag{21}$$

where the entropic function $h(x)$ is defined by equation (10), with the additional proviso $h(1) \equiv 0$ (which renders the function continuous). Recall also that the uncertainty principle reads $\nu_j \geqslant 1$ in terms of the symplectic eigenvalues.

In turn, the symplectic eigenvalues are continuously determined by m symplectic invariants $\{\Delta_d^m\}$, given by the even-order coefficients of the characteristic polynomial of the matrix $\Omega\sigma$ [27]. Let us denote by g the continuous function connecting the invariants to the von Neumann entropy:

$$S = g\left(\Delta_1^m, \ldots, \Delta_m^m\right). \tag{22}$$

Note also that any symplectic invariant Δ_d^m is a homogeneous polynomial of order $2d$ in the entries of γ.

In order to prove the concentration of measure for the entanglement, we will show that the distribution induced by the canonical measure on the space of the symplectic invariants tends, in the thermodynamical limit, to a δ-function centred on their averages (which will also be determined). Through equation (22), this will allow us to infer concentration of measure for the von Neumann entropy (and to determine its asymptotic average as well). Note that this will also imply concentration of measure for any other entropic measure as, for Gaussian states, all such quantities are univocally determined by the symplectic invariants [27, 32].

Let us consider the CM σ of the whole system. Considering the expression (13) for a pure covariance matrix and parametrizing the transformation $O \in K(n)$ in terms of the matrices X and Y such that $(X + \mathrm{i}Y) \equiv U \in U(n)$, according to the isomorphism of equation (5), one has for the entries of σ:

$$\sigma_{jk} = \sum_l \left(X_{jl} X_{kl} z_l^2 + Y_{jl} Y_{kl} z_l^{-2}\right) \qquad \text{for} \quad 1 \leqslant j, k \leqslant n, \tag{23}$$

$$\sigma_{jk} = \sum_l \left(Y_{jl} Y_{kl} z_l^2 + X_{jl} X_{kl} z_l^{-2}\right) \qquad \text{for} \quad n+1 \leqslant j, k \leqslant 2n, \tag{24}$$

$$\sigma_{jk} = \sum_l \left(-X_{jl} Y_{kl} z_l^2 + Y_{jl} X_{kl} z_l^{-2}\right) \qquad \text{for} \quad 1 \leqslant j, (k-n) \leqslant n. \tag{25}$$

As previously remarked, each of the symplectic invariants is a homogeneous polynomial in such entries. Let us now consider, for fixed squeezings z_j's, the averages of such polynomials with respect to the matrices X and Y, distributed according to the Haar measure over $U(n)$.

To work out averages over the Haar measure we shall make use of some basic properties of the integration over the unitary group, derived from simple symmetry arguments. The reader is deferred to [33] for a general discussion of such strategies. Let us also mention that an alternative way for computing Haar integrals with applications to linear optical systems and average entanglement has been discussed in [34]. In particular, since permutation of the indices is clearly a unitary operation, and since the Haar measure is both left- and right-invariant with respect to any unitary operation, the integration over the group only depends on the number and multiplicity of the different left and right indices present, but not on their specific values. Likewise, the measure is invariant under local phase changes (i.e. operations

represented by diagonal matrices with one eigenvalue equal to $e^{i\phi}$ and all the other eigenvalues equal to 1), which implies that all the X_{jk}'s can be swapped with Y_{jk}'s without affecting the values of the integrals. Another consequence of the invariance under local phase changes will be exploited shortly.

Let us consider the average $\overline{\Delta_d^m}$ of the invariant Δ_d^m in the limit $n \to \infty$. This invariant is a polynomial of order $2d$ of elements given by equations (23)–(25), so that each term of the polynomial is a product of sums, of the form

$$\prod_{s=1}^{2d} \sigma_{j_s k_s} = \prod_{s=1}^{2d} \left(\sum_{l=1}^{n} W_{j_s k_s l} \right), \tag{26}$$

where W_{jkl} are polynomials of order two in X and Y depending on the z_l's and determined by equations (23)–(25). Suppose we expand the product occurring on the LHS of equation (26), and sort the contribution of the resulting addenda according to the multiplicity of the different indexes l's, for $1 \leqslant l \leqslant n$. As already remarked, the integral over the Haar measure of the monomials in the matrix elements X_{jl} and Y_{jl}, depends only on such multiplicities. Elementary combinatorial arguments show that the contributions of leading order in n are those for which all the indexes l_k's are different from each other, which encompasses $n!/(n-2d)!$ terms of the resulting sum: all the other terms can be neglected in the thermodynamical limit.

Indeed, only one of the leading terms is non-vanishing. Whenever a term of the sum of equation (25) enters in a monomial where all the l_k's are different, the resulting average is of the form $\overline{X_{j1} Y_{k1} q(X, Y)}$, where the index 1 has been fixed with no loss of generality (due to permutational invariance), while the $q(X, Y)$ is any function of the entries of X and Y where X_{j1} and Y_{k1} do not appear. In terms of the complex unitary matrix $U = X + iY$, one then has

$$4i\overline{X_{j1} Y_{k1} q(X, Y)} = \overline{(U_{j1} + U_{j1}^*)(U_{k1} - U_{k1}^*) q(X, Y)} = 0,$$

because the average has to be invariant under arbitrary left and right phase changes on the indexes j, k and 1, but all the 'phase-invariant' terms $|U_{1j}|^2$ get cancelled out [this holds also for $j = k$, as in that case the factor depending on U_{j1} reduces to $\left(U_{j1}^2 - U_{j1}^{*2}\right)$]. Moreover, if all the l_k's are different, terms coming from the sums of equation (23) are identical to those coming from the sums of equation (24), as X's and Y's indexes can be swapped. Summing up, the leading term as $n \to \infty$ of the average $\overline{\Delta_d^m}$, reads

$$\overline{\Delta_d^m} \simeq c(d, m, n) \sum_{\mathcal{S}_{2d}^n} \overline{\prod_{l \in \mathcal{S}_{2d}^n} \left(z_l^2 + \frac{1}{z_l^2} \right)} = c(d, m, n) \sum_{\mathcal{S}_{2d}^n} \overline{\prod_{l \in \mathcal{S}_{2d}^n} E_l}, \tag{27}$$

where the sum runs over all the possible $2d$-subsets $\mathcal{S}_{2d}^n$ of the first m natural integers (i.e. over all the possible $n!/(n-2d)! \simeq n^{2d}$ combinations of $2d$ integers smaller or equal than n, with no repetitions, equal to all the possible combinations of different indexes), $c(d, m, n)$ is a factor depending on d, m and n which we shall determine shortly, and the symbol $\simeq$ specifies that the equality holds in the thermodynamical limit. Note that, being interested in finite subsystems in the thermodynamical limit, we can always assume $2d < n$. The convenience of the parametrization of the states through the energies $\{E_k\}$ becomes fully apparent in the previous expression. The coefficient $c(d, m, n)$ is easily determined by considering that, setting $E_k = 2 \, \forall k$, one has $\boldsymbol{\sigma} = \mathbb{1}$ regardless of the applied orthogonal transformation, for which the values of the local invariants Δ_d^m are fixed and trivially equal to their averages. This yields

$$\overline{\Delta_d^m} \simeq \binom{m}{d} \frac{\sum_{\mathcal{S}_{2d}^n} \overline{\prod_{l \in \mathcal{S}_{2d}^n} E_l}}{(2n)^{2d}}. \tag{28}$$

Note that the factor n^{-2d} in the normalization is due to the integration over the Haar measure. In fact, such a dependence is common to the average of any monomial of order $2d$ in the matrix elements (when non-null) [33]. Also, let us emphasize that the previous line of thought applies to any polynomial in the entries of $\boldsymbol{\sigma}$. The only specific information about the invariants Δ_d^m entered in imposing the normalization condition, so that all the structure of the invariants could be 'extracted' from a trivial situation (one has $\Delta_d^m = \binom{m}{d}$ for the vacuum [28]).

The same arguments apply to the average $\overline{\left(\Delta_d^m\right)^2}$ (as a squared invariant is just a polynomial of order $4d$ in the entries of $\boldsymbol{\sigma}$), leading to

$$\overline{\left(\Delta_d^m\right)^2} \simeq \binom{m}{d}^2 \frac{\sum_{\mathcal{S}_{4d}^m} \overline{\prod_{l\in\mathcal{S}_{4d}^m} E_l}}{(2n)^{4d}}. \tag{29}$$

To prove the concentration of measure for the invariants, we have to average over the remaining variables $\{E_j\}$. As we have seen, both the canonical and the micro-canonical distributions are i.i.d. in the thermodynamical limit. For the sake of clarity, let us impose the two conditions of invariance under permutations and statistical independence one at a time. The invariance under permutations of the variables $\{E_k\}$ allows one to write equations (28) and (29) as

$$\overline{\Delta_d^m} \simeq \binom{m}{d} \frac{n!\overline{\prod_{k=1}^{2d} E_k}}{(n-2d)!(2n)^{2d}} \simeq \binom{m}{d} \frac{\overline{\prod_{k=1}^{2d} E_k}}{2^{2d}} + O\left(\frac{1}{n}\right),$$
$$\overline{(\Delta_d^m)^2} \simeq \binom{m}{d}^2 \frac{\overline{\prod_{k=1}^{4d} E_k}}{2^{4d}} + O\left(\frac{1}{n}\right),$$

leading to

$$\lim_{n\to\infty} \left(\overline{\Delta_d^m}^2 - \overline{\left(\Delta_d^m\right)^2}\right) = \frac{\binom{m}{d}^2}{16^d} \left(\left(\overline{\prod_{k=1}^{2d} E_k} \right)^2 - \overline{\prod_{k=1}^{4d} E_k} \right). \tag{30}$$

Note that the quenching of the residual terms in this development is due to the integration over the Haar measure of the compact symplectic group, which always results in terms of the order n^{-2j} for polynomials of order $2j$ in X and Y. Finally, the statistical independence of the energies E_j's requires $\overline{\prod_{j=1}^{k} E_j} = \overline{E}^k \,\forall\, k$, such that the previous equation becomes

$$\lim_{n\to\infty} \left(\overline{\Delta_d^m}^2 - \overline{\left(\Delta_d^m\right)^2}\right) = 0. \tag{31}$$

Therefore, the variance of any local symplectic invariant Δ_d^m vanishes in the thermodynamical limit. The previous argument applies to any polynomial of finite order in the entries of $\boldsymbol{\sigma}$. In particular, in the case $2d = 1$, this vanishing of the variance directly applies to the entries of $\boldsymbol{\sigma}$, thus also implying, *a posteriori*, the compliance (already shown in [20] in a more specific background) of the presented measures with the general canonical principle.

To complete our proof and extend the concentration of measure to the local von Neumann entropy, let us consider the measure $\Gamma_n\left(\Delta_1^m, \ldots, \Delta_m^m\right)$ induced, in the m-dimensional real space of the local symplectic invariants $\left\{\Delta_d^m\right\}$, by the proposed canonical measure on n-mode pure Gaussian states. Because of equation (31), such a measure completely concentrates in the average values $\overline{\Delta_d^m}$. Here, we will express this fact by claiming that Γ_n tends to a Dirac delta:

$$\lim_{n\to\infty} \Gamma_n\left(\Delta_1^m, \ldots, \Delta_m^m\right) = \delta\left(\Delta_1^m - \overline{\Delta_1^m}, \ldots, \Delta_m^m - \overline{\Delta_1^m}\right).$$

[see appendix B for a rigorous formulation, showing the emergence of the following limits from equation (31)]. For the von Neumann entropy of the m-mode subsystem S_m, one finds

$$\lim_{n\to\infty} \overline{S_m} = \lim_{n\to\infty} \int \Gamma_n\left(\Delta_1^m, \ldots, \Delta_m^m\right) g\left(\Delta_1^m, \ldots, \Delta_m^m\right) \mathrm{d}^m \Delta_d^m = g\left(\overline{\Delta_1^m}, \ldots, \overline{\Delta_m^m}\right), \tag{32}$$

$$\lim_{n\to\infty} \overline{S_m^2} - \overline{S_m}^2 = 0. \tag{33}$$

The asymptotic average $\overline{S_m}$ can be determined as well. First, let us determine the canonical averages Δ_d^m of the invariants according to equations (28) and (19), finding

$$\lim_{n\to\infty} \Delta_d^m = \binom{m}{d} \left(1 + \frac{T}{2}\right)^{2d}. \tag{34}$$

Next, let us recall the expression of the symplectic invariants in terms of the symplectic eigenvalues $\{\nu_j\}$ [27]:

$$\Delta_d^m = \sum_{\mathcal{S}_d^m} \prod_{k\in\mathcal{S}_d^m} \nu_k^2. \tag{35}$$

Straightforward substitution in equation (35) shows that the values $\nu_k = (1 + T/2)\,, \forall\, 1 \leqslant k \leqslant m$, account for the asymptotic values of the invariants given by equation (34). Thus, for the symplectic eigenvalues $\{\nu_j\}$, one has

$$\lim_{n\to\infty} \overline{\nu_k} = 1 + \frac{T}{2}, \quad \forall\, 1 \leqslant k \leqslant m. \tag{36}$$

In the thermodynamical limit, both the introduced distributions, canonical and micro-canonical, comply with a form of equipartition theorem: an average 'energy' equal to $T/2$ is allotted to each decoupled quadratic degree of freedom. Finally, one has

$$\lim_{n\to\infty} \overline{S} = mh\left(1 + \frac{T}{2}\right). \tag{37}$$

5. Study of the typical entanglement of pure Gaussian states

In the present section, we present a detailed study of the typical bipartite entanglement for Gaussian states with a finite number n of total modes. More specifically, we will study the statistical properties of the entropies of a reduced subsystem of m modes, when the global states are distributed according to the micro-canonical and canonical measures.

As we have remarked in equation (8), the quantity $\det\boldsymbol{\sigma}$ is a proper quantifier of the *mixedness* of Gaussian states, and thus of their entanglement as well, when computed for the reduced subsystem of a pure state. For such an entropic quantity, we were able to derive analytical expressions for the canonical and micro-canonical average and standard deviation, which will be presented in subsection 5.1. In subsection 5.2, we describe a strategy—based on a simple linear minimization—to derive, from such analytical results, exact information about the micro-canonical statistical properties of the actual entropy of entanglement (defined as the von Neumann entropy of the subsystem). In subsection 5.3 we complement this analysis with numerical results, also covering m-mode subsystems.

5.1. Purity of single-mode subsystems

For a single-mode subsystem (corresponding to $m = 1$ in the previous section's notation), we have worked out analytically the average and variance of the inverse squared purity $\mu^{-2} = 1/\,\mathrm{tr}[\varrho^2]^2$, which coincides, for Gaussian states, with the determinant of the reduced covariance matrix γ (maintaining the notation of the previous section). Let us mention that, in this particular instance, such a measure is a perfectly legitimate entanglement quantifier, as it induces, on the set of single-mode states, the same hierarchy as the von Neumann entropy (in fact, both quantities are determined by the single symplectic eigenvalue, see section 2) [36].

Making use of the methods described in [33], we determined the first and second statistical moments over the Haar measure (respectively denoted by μ_H^{-2} and μ_H^{-4}) of the determinant $\det\gamma = \mu^{-2}$, to find

$$\mu_H^{-2} = \sum_{j\neq k} \frac{E_j E_k}{4(n+1)n} + \frac{2}{n+1}, \tag{38}$$

$$\mu_H^{-4} = \frac{1}{16}\frac{(n-1)!}{(n+3)!}\left[\sum_{j\neq k\neq l\neq m} E_j E_k E_l E_m + 8\sum_{j\neq k\neq l} E_j^2 E_k E_l + 12\sum_{j\neq k} E_j^2 E_k^2 \right.$$
$$\left. + (96+16(n-2))\sum_{j\neq k} E_j E_k - 32(n-1)\sum_j E_j^2 + 128n(n-1) + 384n\right], \tag{39}$$

where the averages over the variables $\{E_j\}$ are still to be worked out, according to the chosen distribution.

The canonical and micro-canonical averages are then straightforward to compute. In the canonical instance, the average and second moment μ_c^{-2} and μ_c^{-4} read

$$\mu_c^{-2} = \frac{1}{4}\frac{n-1}{n+1}(T^2+4T) + 1, \tag{40}$$

$$\mu_c^{-4} = \frac{1}{16}\frac{n!(n-1)}{(n+3)!}[(n^2+11n+22)T^4 + 8(n^2+8n+6)T^3 + 8(3n^2+15n+10)T^2$$
$$+ 32(n+3)(n+2)T] + 1. \tag{41}$$

Whereas, in the micro-canonical case, defining $\tilde{E} \equiv (E-2n)$, one gets

$$\mu_{mc}^{-2} = \frac{(n-1)}{4(n+2)(n+1)^2}(\tilde{E}^2 + 4(n+2)\tilde{E}) + 1, \tag{42}$$

$$\mu_{mc}^{-4} = \frac{(n!)^2(n-1)}{16(n+4)!(n+3)!}((n^2+11n+22)\tilde{E}^4 + 8(n+6)(n+4)(n+1)\tilde{E}^3$$
$$+ 8(n+4)(n+3)(3n^2+15n+10)\tilde{E}^2 + 32(n+4)(n+3)^2(n+2)^2\tilde{E}) + 1. \tag{43}$$

Note that the maximal μ_M^{-2} for given energy $E = \tilde{E} + 2n$ is (see appendix C)

$$\mu_M^{-2} = \frac{(\tilde{E}+4)^2}{16}. \tag{44}$$

Whilst restricted to a particular quantifier and to a single mode, these analytical results display the most relevant statistical features of the entanglement of pure Gaussian states. The micro-canonical mean μ_{mc}^{-2} is monotonically increasing with E for fixed n and, for $n > 2$, monotonically decreasing with n for given E (a 'finite size' effect shows up for $\tilde{E} \leqslant 10$, where μ_{mc}^{-2} increases in going from 2 to 3 modes). This can be promptly explained as more available energy generally allows for higher entanglement, while the presence of more modes 'drains' energy away to establish correlations which do not involve the particular chosen mode. On the other hand, the canonical average entanglement is monotonically increasing with both temperature and number of modes. This behaviour is encountered also for the micro-canonical entanglement with given maximal total energy *per mode*, which is, even for small n, closely akin to the canonical ensemble (upon replacing $\tilde{E}/n$ with T). The increase of the average canonical entanglement with increasing number of modes but fixed temperature is a non trivial, purely 'geometric' effect, due to the average over the Haar measure $\mathrm{d}\mu_H(\vartheta)$ of the compact variables ϑ. An analogous increase is in fact observed assuming a given, fixed value for the variables z_j's and averaging only over the compact variables ϑ: as the number of total modes increases, a given mode has more possibilities of getting entangled, even keeping a fixed mean energy per mode.

As for the standard deviations, which are straightforward to derive from the expressions above, they are generally increasing with total energy and temperature for fixed total number of modes (as more energy allows for a broader range of entanglement). Significantly, these partial analytical results clearly show the arising of the concentration of measure around a thermal average. Both for the canonical case and for the micro-canonical one with $\tilde{E} = nT$ the standard deviation decreases with increasing number of modes n, falling to zero asymptotically (after transient 'finite size' effects, for very small n). Moreover, in the micro-canonical instance, the thermal average of concentration is generally very distant, even for relatively small n, from the allowed maximum of equation (44) (which clearly diverges in the thermodynamical limit): e.g., for $\tilde{E} = 10n$ one has that the average μ_{mc}^{-2} is, respectively 16.5 and 257.1 standard deviations away from the maximal value μ_M^{-2} for $n = 5$ and $n = 20$. Such a distance increases monotonically with the total number of modes. This peaked concentration for finite n will be exploited in the next subsection to obtain strict bounds on the average von Neumann entropy of entanglement of single-mode subsystems.

5.2. Estimating the micro-canonical mean entropy of entanglement

In the previous section, we have focused on the inverse square purity μ^{-2}, because it can be readily described in terms of the covariance matrix's entries, thus allowing one to determine analytical expressions for the entanglement's statistics. Let us stress once more that, for pure states, such a quantity is a legitimate entanglement measure, as it is a monotone strictly related to the so-called linear entropy, given by $1 - \mu$. Moreover, for single-mode Gaussian states, the linear entropy is, as we have already remarked, monotonically determined by the von Neumann entropy alone. However, the proper 'entropy of entanglement' (given by the von Neumann entropy of the reduced density matrix) is endowed with a clear operational meaning for pure states (as it corresponds to the rate of singlets distillable from the state in the asymptotic limit of infinite copies [35]). It is thus highly desirable to obtain qualitative and quantitative information about the statistical properties of such a quantity under our measures. Clearly, the findings of section 4 already provide us with such results in the thermodynamical limit, where the von Neumann entropy concentrates, with vanishing variance, around the value set out in equation (37).

Here, we shall address the entropy of entanglement of a single-mode subsystem in a system with a finite total number of modes. Exploiting the fact that the von Neumann entropy S is continuously determined by μ^{-2} as [36]

$$S(\mu) = h(\mu^{-1}) \tag{45}$$

(under the additional prescription $S(1) \equiv 0$), we will show how upper and lower bounds on the micro-canonical average S_{mc} can be derived. The reason why we focus on the micro-canonical measure is that, imposing a bound on the energy, it involves a maximal value for the entanglement (as discussed in appendix C), which is another key ingredient we are going to use. Extending the method we are presenting to the canonical measure is possible, at the price of introducing an approximation to neglect the 'tail' of the canonical distribution in the space of the entanglement (certainly feasible with reasonable resources for small enough temperatures). As we will see, such bounds, which we know to become increasingly close with increasing n because of concentration, can be quite tight even for relatively small n, thus providing precise information about the average entropy of entanglement.

For the sake of readability, we will from now on set $a := \mu^{-2}$. For a given number n of modes and total energy E in the system, the maximum value $a_{\max}$ for the inverse squared purity of any fixed mode is given by equation (44), while the minimum value is always (for $n > 2$) given by $a_{\min} = 1$, corresponding to the case where the mode is decoupled from the remainder of the system. Our measure on Gaussian states will induce a probability distribution ν on the interval $[a_{\min}, a_{\max}]$. While we do not know ν directly, we are aware of two of its properties, namely the averages a_{mc} and ${a^2}_{mc}$ with respect to it, which we computed in the previous subsection. Our approach is going to be as follows: to obtain an upper bound for S_{mc} we maximize S_{mc} over all probability distributions ν' which produce the right averages for a and a^2. There is a technical obstacle to the pursuit of this programme: the set of all probability distributions on the interval is infinite dimensional. To circumvent this problem, we will partition $[a_{\min}, a_{\max}]$ into M equally sized sub-intervals ($M \in \mathbb{N}$ being an arbitrary parameter) and consider 'discretized' probability distributions, which are constant over these subintervals. This is going to be done in such a manner that the resulting bounds are valid for any finite M, and not only in the limit of $M \to \infty$. Also, it will turn out that all optimizations can be cast into the form of linear programmes. This excludes the occurrence of local minima and implies that the obtained bounds will hold rigorously.

More explicitly, fix an $M \in \mathbb{N}$. The kth sub-interval will be $[l(k), l(k+1)]$, where $l(k) \equiv a_{\min} + k\frac{a_{\max}-a_{\min}}{M}$ is its leftmost point. Let $\nu_k := \nu([l(k), l(k+1)]$ be the measure of the kth sub-interval. Clearly, one has

$$a_{mc} = \int_{a_{\min}}^{a_{\max}} a \, \mathrm{d}\nu(a) \leqslant \sum_{k=0}^{M-1} l(k+1)\nu_k, \qquad a_{mc} \geqslant \sum_{k=0}^{M-1} l(k)\nu_k, \tag{46}$$

$$a_{mc}^2 \leqslant \sum_{k=0}^{M-1} l(k+1)^2\nu_k, \qquad a_{mc}^2 \geqslant \sum_{k=0}^{M-1} l(k)^2\nu_k. \tag{47}$$

Furthermore, because $S(a)$ is monotonously increasing in a, we have

$$S_{mc} = \int_{a_{\min}}^{a_{\max}} S(a) \, \mathrm{d}\nu(a) \leqslant \sum_{k=0}^{M-1} S(l(k+1))\nu_k. \tag{48}$$

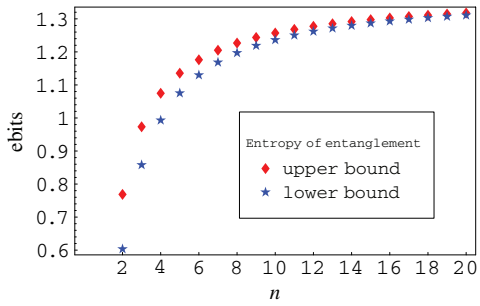

Figure 1. Upper and lower bounds to the micro-canonical average von Neumann entropy of entanglement of one mode as a function of the number of total modes n. The maximal energy was assumed to be $4n$ (in units of $\hbar\omega/4$). To compute the bounds, the method described in subsection 5.2 has been employed. The parameter M was chosen to be 10 000.

Now let $P \subset \mathbb{R}^M$ be the set of probability distributions which are constant on the M subintervals and compatible with equations (46) and (47). Certainly, the discrete distribution given by the ν_k is an element of P and we can thus deduce from equation (48) that

$$S_{mc} \leqslant \sup_{\nu' \in P} \sum_{k=0}^{M-1} S(l(k+1))\nu'_k. \tag{49}$$

Hence, the following linear programme with variables $\nu' \in \mathbb{R}^M$ yields an upper bound for S_{mc}:

$$\text{maximize} \sum_k S(l(k+1))\nu'_k$$

$$\text{subject to } a_{mc} \leqslant \sum_k l(k+1)\nu'_k, \qquad a_{mc} \geqslant \sum_k l(k)\nu'_k,$$

$$a_{mc}^2 \leqslant \sum_k l(k+1)^2\nu'_k, \qquad a_{mc}^2 \geqslant \sum_k l(k)^2\nu'_k,$$

$$\sum_k \nu'_k = 1.$$

A lower bound is found by a completely analogous minimization programme.

Figure 1 depicts the results of the programme laid out above for a specific choice of energy per mode. Quite remarkably, already for a small number of modes, one can give a fairly precise value for the average micro-canonical entropy of entanglement, thus complementing the asymptotic statement of section 4. Let us also mention that the restriction imposed by the knowledge of the second moments is crucial in rendering the bounds so tight (as opposed to the use of the averages alone).

5.3. Numerical results on the entropy of entanglement

Unfortunately, the analytical and exact methods illustrated above cannot be easily extended to more complicated situations. However, our measures are well suited to be numerically investigated by direct sampling (the Haar measure-distributed symplectic orthogonal can be reproduced by generating unitary matrices from the Gaussian unitary ensemble [37] and then by translating them according to equation (5)).[4] This allows for the investigation of the statistical properties of the actual entropy of entanglement for varying values of m, n and of the parameters of the measures E or T, according to the setting in question.

[4] The MATLAB code we made use of is available at www.imperial.ac.uk/quantuminformation.

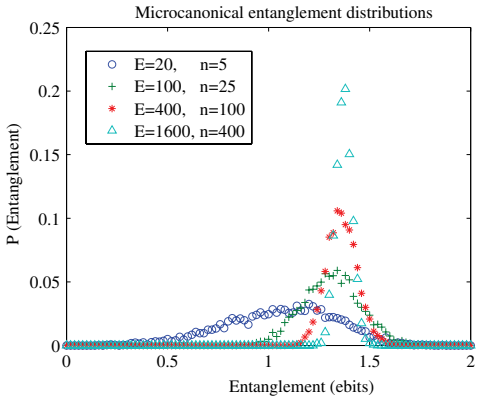

Figure 2. Histograms of microcanonical entanglement distributions. $m = 1$, and 5000 samples were taken for each distribution. The concentration of measure with increasing n is apparent.

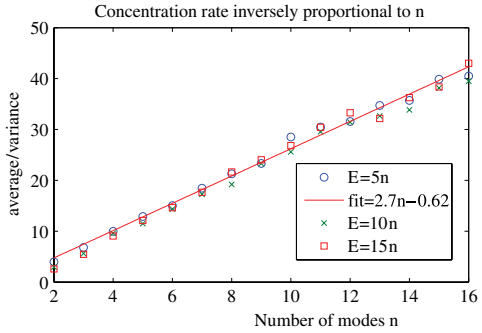

Figure 3. Numerical evidence that the ratio variance/average, for a microcanonical measure on n-mode states with maximal energy E, is proportional to $1/n$ and, surprisingly, that the proportionality factor is independent of E/n. A single-mode subsystem was considered ($m = 1$); 1500 states were sampled for each data point.

In figure 2 a sequence of numerically generated microcanonical probability distributions for the entanglement of a single mode are plotted, unambiguously showing the concentration of measure for the von Neumann entropy at small n. Note that, for low enough energies in the microcanonical case, even for small n—well before the onset of thermodynamical concentration of measure around the finite thermal average—the entanglement of pure Gaussian states distributes around values generally distant from the finite allowed maximum: e.g., for $m = 1$ and $E = 10n$, the difference between the maximum and the average $\overline{S}$ is, respectively, 4.0 and 13.6 standard deviations for $n = 5$ and $n = 20$.

Furthermore, for the micro-canonical case, our investigation through sampling indicates that the ratio between variance and average is inversely proportional to n for finite n and, rather surprisingly, that the proportionality factor is independent of E/n (see figure 3).

When more than one mode is addressed (i.e. when $m > 1$), numerics show that the average canonical entanglement is roughly proportional to the number of local modes m. The same linear approximation carries over to the micro-canonical case for $E \gg n$ and $m \ll n$, upon substituting E/n for T. Clearly, such an approximate proportionality, rigorously true in the thermodynamical limit [see equation (37)], is better satisfied for increasing number of total modes n, but provides good estimates (up to some percent) already for $n \approx 30$ and small enough m, as is shown in figure 4(a), where a clear subadditive behaviour for

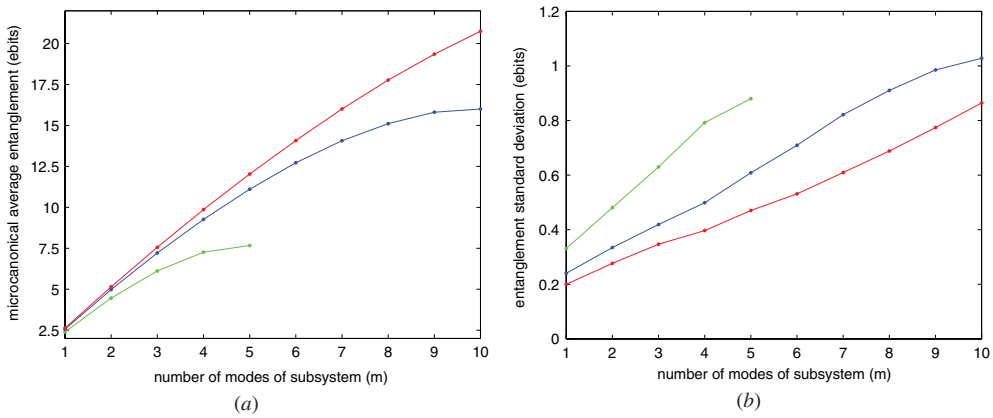

Figure 4. Microcanonical average entropy of entanglement (a) and standard deviation of the entropy of entanglement's distribution (b) for $E = 10n$, as a function of the number of modes m of the reduced subsystem. On the left (a), the upper curve (red online) refers to $n = 30$, the middle curve (blue online) to $n = 20$ and the lower curve (green online) to $n = 10$. On the right (b), the upper curve (green online) refers to $n = 10$, the middle curve (blue online) to $n = 20$ and the lower curve (red online) to $n = 30$. Only the range $1 \leqslant m \leqslant 5$ was considered for $n = 10$. Data were obtained from samples of 5000 states.

$m \lesssim n$ is also apparent. Also, let us point out that the concentration of measure for the entanglement distribution clearly shows up at finite n for $m > 1$ as well. For instance, for a microcanonical distribution with $m = 5, n = 20$ and $E = 200$, one finds that the average is 11.4 standard deviations away from the allowed maximum. Also, figure 4(b) shows how the standard deviation increases approximately proportionally to m and decreases for increasing n for $m \neq 1$ as well.

5.4. Concentration of measure for infinite subsystems

So far, we have considered the concentration of measure occurring for a finite subsystem with a fixed number of modes m. Let us now briefly turn to the case of a fixed *'ratio'* of subsystems, where $m/n > 0$ in the thermodynamical limit. Clearly, in such a case, the subsystem as well comprises infinitely many modes.

The analytical reasoning of section 4 can be readily adapted to this case and shows, not surprisingly, that the average $\overline{S}$ of the von Neumann entropy diverges in the thermodynamical limit. In this case though, analogous arguments imply that the variance $\overline{S^2} - \overline{S}^2$ of the von Neumann entropy diverges as well. Still, numerics strongly suggest the occurrence of a weaker form of concentration, namely

$$(\overline{S^2} - \overline{S}^2)/\overline{S}^2 \simeq 0. \tag{50}$$

This behaviour, unambiguously supported by the numerical analysis in both the micro-canonical and canonical instances, extends to the general case of large m the evidence depicted in figure 3 (according to which the ratio between variance and average is inversely proportional to n).

6. Conclusions and outlook

The approach we have introduced here tames the divergence due to the infinite dimension of the Hilbert spaces of continuous variable systems and allows one to introduce a well defined

notion of 'typical' continuous variable entanglement. Our exhaustive analytical and numerical study shows that such a typical entanglement concentrates sharply around a thermal average, even for a relatively small number of modes. In the micro-canonical case, where the upper bound to the energy implies the existence of an absolute upper bound for the entanglement, the average entanglement turns out to be different and well separated from the maximal value, by many standard deviations, even for very small number of modes. Under such 'heavily thermodynamical' prescriptions—as are those we have adopted to construct the canonical and micro-canonical measures—equipartition prevents the entanglement from reaching the allowed maximum, evenly spreading the correlations between all the modes.

Our measures, being compliant with the general canonical principle, should be suitable to the description of dynamical situations [12, 21], also involving randomized interactions [7]. A systematic study of such processes—notably restricting to two-mode interactions, in the spirit of [7]—where the two measures defined here would arise as stationary distributions, is the next direction to pursue in the present line of research.

This line of enquiry may also contribute to our understanding of why objects typically appear classical despite being governed, as is normally assumed, by quantum mechanics. In particular, this and other related works, such as [6, 14, 12], imply that, always in a particular sense, most global pure quantum states have the property that a local state of a comparatively small party, obtained by tracing over the larger party, will be highly mixed and accordingly have little or no quantum correlations. This is at least one necessary ingredient for the system to appear classical. The work here indicates that these arguments can be applied in the continuous variable setting too. It would be interesting to combine, compare and further develop existing approaches in order to explain the emergence of apparent classicality in realistic models of physical systems. As a further investigation, one could consider the extension of analogous micro-canonical and canonical measures on general finite-dimensional states [21]. One could then distinguish, to a greater detail, the features induced by the chosen thermodynamical setting from the ones proper to the Gaussian continuous variable scenario.

Finally, let us mention that the presented framework may be suitably extended to more general, *mixed* Gaussian states. In fact, the Williamson decomposition of a generic covariance matrix [equation (7)], together with the Euler decomposition given by equation (11), suggest that, to encompass mixed states as well, one has to add only another set of compact variables ϑ', plus the symplectic eigenvalues of the global mixed states $\{\nu_j\}$. The presented approach thus also paves the way for the definition of more general measures on the whole set of Gaussian states.

Acknowledgments

We thank R Oliveira, J Eisert and K Audenaert for discussions. OD was funded by the Institute for Mathematical Sciences of Imperial College. MBP was funded by the EPSRC QIPIRC, The Leverhulme Trust, EU Integrated Project QAP, and the Royal Society. AS is a Marie Curie Fellow.

Appendix A. A Baker–Campbell–Hausdorff formula for quadratic polynomials in the canonical operators

For the sake of completeness and self-consistency, we present here a proof of equation (4). To this end, we will follow a strategy customarily adopted in the derivation of Baker–Campbell–Hausdorff–like relations (see, e.g., [29]). Let us thus define the operator

$\hat{G}_{A,b}(\xi) \equiv \mathrm{e}^{\mathrm{i}\xi(\hat{R}^{\mathsf{T}}A\hat{R}+\hat{R}^{\mathsf{T}}b)}$, where ξ is a real variable. Note that the CCR straightforwardly imply

$$[\hat{R}^{\mathsf{T}}A\hat{R}, \hat{R}^{\mathsf{T}}b] = 4\mathrm{i}A\Omega b. \tag{A.1}$$

We start by assuming the following ansatz, which we will prove shortly,

$$\hat{G}_{A,b}(\xi) \equiv \mathrm{e}^{\mathrm{i}\xi(\hat{R}^{\mathsf{T}}A\hat{R}+\hat{R}^{\mathsf{T}}b)} = \mathrm{e}^{\mathrm{i}f(\xi)(\hat{R}^{\mathsf{T}}A\hat{R})}\mathrm{e}^{\hat{R}^{\mathsf{T}}(M(\xi)b)}, \tag{A.2}$$

where $f(\xi)$ is a scalar real function, whereas $M(\xi)$ is a $2n \times 2n$ matrix depending continuously on ξ. Differentiating both sides with respect to ξ, we find (differentiation is denoted by $'$):

$$-\mathrm{i}\frac{\mathrm{d}\hat{G}_{A,b}}{\mathrm{d}\xi}(\xi) = (\hat{R}^{\mathsf{T}}A\hat{R} + \hat{R}^{\mathsf{T}}b)\hat{G}_{A,b}(\xi) \tag{A.3}$$

$$= (f'(\xi)\hat{R}^{\mathsf{T}}A\hat{R} + \hat{R}^{\mathsf{T}}\mathrm{e}^{\mathrm{i}f(\xi)(\hat{R}^{\mathsf{T}}A\hat{R})}M'(\xi)b\,\mathrm{e}^{-if(\xi)(\hat{R}^{\mathsf{T}}A\hat{R})})\hat{G}_{A,b}$$

$$= (f'(\xi)\hat{R}^{\mathsf{T}}A\hat{R} + \hat{R}^{\mathsf{T}}\mathrm{e}^{-4f(\xi)A\Omega}M'(\xi)b)\hat{G}_{A,b}, \tag{A.4}$$

where we have made use of equation (A.1) in the last step. Equating (A.3) and (A.4) yield the following systems of differential equations:

$$f'(\xi) = 1, \tag{A.5}$$

$$M'(\xi) = \mathrm{e}^{4f(\xi)A\Omega}, \tag{A.6}$$

with initial conditions $f(0) = 0$ and $M(0) = 0$. This system always admits an analytical solution, given in general by $f(\xi) = \xi$ and $M(\xi) = \int_0^{\xi} \mathrm{e}^{4\xi' A\Omega}\mathrm{d}\xi'$, whose value in $\xi = 1$ gives the matrix $M = M(1)$, thus proving the validity of equation (4). If A is invertible, M is simply given by

$$M = \tfrac{1}{4}\Omega A^{-1}(\mathbb{1}_{2n} - \mathrm{e}^{4A\Omega}). \tag{A.7}$$

Appendix B. Asymptotic concentration of measure

In this appendix, we show how the concentration of the measure Γ_n in the space of the symplectic invariants follows from equation (31). In doing so, we will provide the reader with a formal derivation of equations (32) and (33).

Let us consider the m-dimensional real space Δ^m of the vectors of symplectic invariants $\Delta \equiv \left(\Delta_1^m, \ldots, \Delta_m^m\right)$, endowed with the usual Euclidean norm $\|\cdot\|$. Let D_ε be a spherical ball of radius ε centred in $\overline{\Delta} \equiv \left(\overline{\Delta_1^m}, \ldots, \overline{\Delta_m^m}\right)$: $D_\varepsilon = \{\Delta : \|\Delta - \overline{\Delta}\| \leqslant \varepsilon\}$. Let R_ε be the complement of D_ε: $R_\varepsilon = \{\Delta : \|\Delta - \overline{\Delta}\| > \varepsilon\}$. Recall that Γ_n stands for the (normalized) measure induced on the space Δ^m by the canonical measure of n-mode pure Gaussian states. Also, $\mathrm{d}\Gamma_n(\Delta)$ will stand for the infinitesimal element of such a measure. Let us remark that equation (31), holding $\forall\, d$, implies

$$\lim_{n\to\infty}(\overline{\|\Delta\|^2} - \|\overline{\Delta}\|^2) = \lim_{n\to\infty}\overline{\|\Delta - \overline{\Delta}\|^2} = 0. \tag{B.1}$$

Our first aim is deriving a rigorous formulation of 'concentration of measure', i.e.:

$$\forall\,\varepsilon > 0 \qquad \text{and} \qquad \forall\,\xi > 0, \qquad \exists\,\tilde{n} \mid \forall\, n > \tilde{n} : \Gamma_n(R_\varepsilon) < \xi. \tag{B.2}$$

To this aim suppose, *ad absurdum*, that the latter statement did not hold. Then, $\forall\, n, \exists\varepsilon, \xi > 0$ and $n_0 > n$ such that one has $\Gamma_{n_0}(R_\varepsilon) > \xi$. But this would imply that, $\forall\, n, \exists\, n_0 > n$ such that

$\overline{\|\Delta - \overline{\Delta}\|^2} \geqslant \xi\varepsilon^2$, which would overtly contradict equation (B.1). Because of normalization, the following equation, complementary to the previous one, holds as well

$$\forall\, \varepsilon > 0 \qquad \text{and} \qquad \forall\, \xi > 0, \qquad \exists\, \tilde{n} \mid \forall\, n > \tilde{n} : \Gamma_n(D_\varepsilon) > (1 - \xi). \tag{B.3}$$

Together with equation (B.1), the two previous statements entail

$$\forall\, \varepsilon > 0 \qquad \text{and} \qquad \forall\, \xi > 0, \qquad \exists\, \tilde{n} \mid \forall\, n > \tilde{n} : \int_{R_\varepsilon} \|\Delta\|^2 \, \mathrm{d}\Gamma_n(\Delta) < \xi. \tag{B.4}$$

The consequences of these facts for the local von Neumann entropy can be easily derived by exploiting the following simple properties. As apparent from equation (35), $\Delta_d^m \geqslant \left(\nu_j^m\right)^2$, $\forall\, j$ and $\forall\, d$: any symplectic invariant is larger than any symplectic eigenvalue. Moreover, for the function $h(x)$—defining the von Neumann entropy according to equation (9)—one has $x^2 > h(x)$ for $x \geqslant 1$ (as is the case for the symplectic eigenvalues, lower bounded by 1 because of the uncertainty principle). This results into $\|\Delta\|^2 \geqslant S$. One can also show that $\|\Delta\|^2 \geqslant S^2$. Therefore, bounds analogous to (B.4) hold for the integrals over R_ε of S and S^2 as well. Let us also note that the function $g(\Delta)$, relating the symplectic invariants to the von Neumann entropy, is certainly continuous (as it relates a continuous function of the eigenvalues of a strictly positive matrix to the coefficients of the characteristic polynomial of the matrix). Putting everything together we find

$$\forall\, \xi > 0, \qquad \exists\, \tilde{n} \text{ s.t. } \forall\, n > \tilde{n} : (g(\overline{\Delta}) - \xi)(1 - \xi) + \xi \leqslant \overline{S} \leqslant (g(\overline{\Delta}) + \xi) + \xi, \tag{B.5}$$

which is just equivalent to equation (32). In the previous inequalities the continuity of $g(\Delta)$ has been invoked and the integral giving the average $\overline{S}$ has been decomposed into an integral over D_ε and an integral over R_ε. An identical argument holds for the average $\overline{S^2}$, which can be shown to converge to $g(\overline{\Delta})^2$, thus proving equation (33) as well and completing our treatment.

Appendix C. Maximal entanglement for given energy

We derive here an expression for the maximal value of the entanglement (44) of pure Gaussian states for given energy (under a generic $m+n$ mode bipartition), by adopting an explicit phase space approach. To begin with, let us remark that any pure Gaussian state of $m+n$ modes can be reduced, by local (with respect to the $m+n$ mode bipartition) symplectic operations, into the tensor product of m two-mode squeezed states and of $n-m$ uncorrelated vacua (here we assume, without loss of generality, $m \leqslant n$) [38, 39]. Now, the local reduction of such a state pertaining to the m-mode system is a Gaussian state with CM in Williamson form. We will now prove that, amongst the CM's with the same symplectic spectrum, the Williamson form is the one for which the second moments' contribution to the energy $E = \operatorname{tr}\boldsymbol{\sigma}$ is minimal. To this aim, let us recall that a generic CM $\boldsymbol{\sigma}$ with Williamson form $\boldsymbol{\nu}$ (and symplectic spectrum given by $\{\nu_j$, for $1 \leqslant j \leqslant m\}$ can be written as

$$\boldsymbol{\sigma} = O'^{\mathsf{T}} Z O^{\mathsf{T}} \boldsymbol{\nu} O Z O', \tag{C.1}$$

where the Euler decomposition of a generic symplectic transformation has been applied. Clearly, the orthogonal transformation O' do not affect the energy and can be neglected in what follows. As for the other terms, let us define $Z' = \operatorname{diag}(z_1, \ldots, z_m)$, $\nu' = \operatorname{diag}(\nu_1, \ldots, \nu_m)$ and X and Y such that $(X + \mathrm{i}Y) \in U(m)$ [by virtue of the isomorphism of equation (5)], to obtain

$$\begin{aligned} \operatorname{tr}\boldsymbol{\sigma} &= \operatorname{tr}\left[(Z'^2 + Z'^{-2})(X^{\mathsf{T}}\nu' X + Y^{\mathsf{T}}\nu' Y)\right] \geqslant \\ &\geqslant 2\operatorname{tr}\left[X^{\mathsf{T}}\nu' X + Y^{\mathsf{T}}\nu' Y\right] = 2\operatorname{tr}\nu' = \operatorname{tr}\boldsymbol{\nu}, \end{aligned}$$

where the inequality ensues from a basic property of the trace of a product of positive matrices (see [40] and note that, obviously, the eigenvalues of $(Z'^2 + Z'^{-2})$ are larger than 2) and from the fact that the transformation parametrized by X and Y is orthogonal and preserves the trace. The state achieving maximal entanglement for given energy E is thus a tensor product of m two-mode squeezed states (being the state with minimal energy for given entanglement). The von Neumann entropy S of the m-mode reduction of such a state is given by equation (9), with the local symplectic eigenvalues $\{\nu_j\}$ subject to the constraint $E = 4\sum_{j=1}^{m} \nu_j + 2(n-m)$. Because of the concavity of $h(x)$, the optimal choice of ν_j's, maximizing the local entropy, is simply given by $\nu_j = \frac{E-2(n-m)}{4m} \forall\, j$, in compliance with the previous constraint. Finally, the maximal von Neumann entropy $S_{\max}(m, n, E)$ of an m-mode reduction of a $(m+n)$-mode pure Gaussian state with $m \leqslant n$ and total energy E is

$$S_{\max}(m, n, E) = mh\left(\frac{E - 2(n-m)}{4m}\right). \tag{C.2}$$

Note that this expression diverges in the thermodynamical limit. The corresponding minimal purity is given by equation (44).

References

[1] Plenio M B and Virmani S 2007 *Quant. Inf. Comput.* **7** 1
[2] Lubkin E 1978 *J. Math. Phys* **19** 1028
[3] Lloyd S and Pagels H 1988 *Ann. Phys.* **188** 186
[4] Page D N 1993 *Phys. Rev. Lett.* **71** 1291
[5] Foong S K and Kanno S 1994 *Phys. Rev. Lett.* **72** 1148
[6] Hayden P, Leung D W and Winter A 2006 *Commun. Math. Phys.* **265** 95
[7] Oliveira R, Dahlsten O C O and Plenio M B 2007 *Phys. Rev. Lett.* **98** 130502
[8] Dahlsten O C O, Oliveira R and Plenio M B 2007 *J. Phys. A: Math. Theor.* **40** 8081–108
[9] Smith G and Leung D W 2006 *Phys. Rev.* A **74** 062314
[10] Dahlsten O C O and Plenio M B 2006 *Quant. Inf. Comput.* **6** 527
[11] Eisert J, Simon C and Plenio M B 2002 *J. Phys. A: Math. Gen.* **35** 3911
[12] Gemmer J, Otte A and Mahler G 2001 *Phys. Rev. Lett.* **86** 1927
[13] Gemmer J, Michel M and Mahler G 2005 *Quantum Thermodynamics—Emergence of Thermodynamic Behavior Within Composite Quantum Systems (Springer Lecture Notes in Physics 657)* (Berlin: Springer)
[14] Popescu S, Short A J and Winter A 2006 *Nature Phys.* **2** 754
[15] Furusawa A, Sørensen J L, Braunstein S L, Fuchs C A, Kimble H J and Polzik E S 1998 *Science* **282** 706
[16] Silberhorn Ch, Lam P K, Wei O, König F, Korolkova N and Leuchs G 2001 *Phys. Rev. Lett.* **86** 4267
[17] Laurat J, Keller G, Oliveira-Huguenin J A, Fabre C, Coudreau T, Serafini A, Adesso G and Illuminati F 2005 *J. Opt. B: Quantum Semiclass. Opt.* **7** S577
[18] Braunstein S L and van Loock P 2005 *Rev. Mod. Phys.* **77** 513
[19] Eisert J and Plenio M B 2003 *Int. J. Quant. Inf.* **1** 479
[20] Serafini A, Dahlsten O C O and Plenio M B 2007 *Phys. Rev. Lett.* **98** 170501
[21] Verhulst T and Naudts J 2007 *J. Phys. A: Math. Gen.* **40** 2475
[22] Folland G B 1989 *Harmonic Analysis in Phase Space* (Princeton: Princeton University Press)
[23] Arvind, Dutta B, Mukunda N and Simon R 1995 *Pramana J. Phys.* **45** 471 (*Preprint* quant-ph/ 9509002)
[24] Williamson J 1936 *Am. J. Math.* **58** 141
[25] Simon R, Chaturvedi S and Srinivasan V 1999 *J. Math. Phys.* **40** 3632
[26] Braunstein S L 2005 *Phys. Rev.* A **71** 055801
[27] Serafini A 2006 *Phys. Rev. Lett.* **96** 110402
[28] Serafini A 2007 *J. Opt. Soc. Am.* B **24** 347
[29] Barnett S M and Radmore P M 1997 *Methods in Theoretical Quantum Optics* (Oxford: Clarendon)
[30] Braunstein S L, Fuchs C A and Kimble H J 2000 *J. Mod. Opt.* **47** 267
[31] Hammerer K, Wolf M M, Polzik E S and Cirac J I *Phys. Rev. Lett.* **94** 150503
[32] Adesso G, Serafini A and Illuminati F 2004 *Phys. Rev.* A **70** 022318
[33] Aubert S and Lam C S 2003 *J. Math. Phys.* **44** 6112
[34] Gross D, Audenaert K and Eisert J 2006 *Preprint* quant-ph/0611002

[35] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 *Phys. Rev.* A **53** 2046
[36] Serafini A, Illuminati F and De Siena S 2004 *J. Phys. B: At. Mol. Opt. Phys.* **37** L21
[37] Metha M L 1991 *Random Matrices* (San Diego: Academic)
[38] Botero A and Reznik B 2003 *Phys. Rev.* A **67** 052311
[39] Giedke G, Eisert J, Cirac J I and Plenio M B 2003 *Quant. Inf. Comput.* **3** 211
[40] Bathia R 1997 *Matrix Analysis* (New York: Springer) p 78